

Universal differential calculus on ternary algebras

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Abstract

General concept of *ternary algebras* is introduced in this article, along with several examples of its realization. Universal envelope of such algebras is defined, as well as the concept of *tri-modules* over ternary algebras. The universal differential calculus on these structures is then defined and its basic properties investigated.

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1 Introduction

We start introducing some notation and conventions. Throughout this article, we shall work in the category of vector spaces over a field \mathbb{K} , which in our case, for simplicity, is assumed to be the field of real or complex numbers. This means that all objects considered here are linear spaces, all mappings are \mathbb{K} -linear mappings, the tensor product \otimes is a shortcut for $\otimes_{\mathbb{K}}$. Algebras will be generically denoted by \mathcal{A} , and the modules will be denoted by \mathcal{M} .

In this letter we are interested in *ternary algebras*, i.e. linear spaces over \mathbb{K} endowed with a trilinear associative composition law. More general structures of this type, called *n-ary algebras* have been studied elsewhere ([1], [4], [6], [8], [10], [14], [15], [18], [20]), and it has been shown that many familiar notions from the theory of usual (i.e. "binary") algebras, such as nilpotency, solvability, simplicity algebras etc., can be quite naturally generalized to the *n*-linear case.

Our attention will be focused on particular properties of *ternary* algebras, including the relations existing between general ternary algebras or ternary algebras of particular types, and trivial ternary algebras induced by the associative law in ordinary algebras, which then play a role similar to the role played by associative algebras with respect to the classical non-associative Lie algebras. Next, we shall define the analog of *modules* over ordinary algebras, which will be called *tri-modules* in the present case.

Finally, we shall define the *derivations* of ternary algebras and show in several examples how such differential ternary algebras can be realized. General construction of the universal envelope for associative ternary algebras and the universal differential calculus will also be presented.

2 Ternary algebras and tri-modules

2.1 Associative ternary algebras

By *ternary* (associative) algebra $(\mathcal{A}, [\])$ we mean a linear space \mathcal{A} (over a field \mathbb{K}) equipped with a linear map $[\] : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ called a (ternary) multiplication (or product), which satisfies the following strong associativity condition :

$$[[abc]de] = [a[bcd]e] = [ab[cde]]$$

Weaker versions of ternary associativity, when only one of the above identities is satisfied, can be called *left* (respectively, *right* or *central*) associativity.

We look at associative ternary algebras as a natural generalization of binary one: If (\mathcal{A}, \cdot) is the usual (binary, associative) algebra then an induced ternary multiplication can be, of course, defined by $[abc] = (a \cdot b) \cdot c = a \cdot (b \cdot c)$. In what follows, such ternary algebras will be called trivial; from now on we shall study exclusively non-trivial ternary algebras. It is known that unital ternary algebras are trivial. Later on we shall show that any finitely generated ternary algebra is a ternary subalgebra of some trivial ternary algebra, which is a ternary generalization of Ado's theorem for finite-dimensional Lie algebras.

As we have already mentioned, many notions known in the binary case can be directly generalized to the ternary case. For example, the notion of ternary \star -algebra is defined by $[abc]^* = [c^*b^*a^*]$, where the star operation $* : \mathcal{A} \rightarrow \mathcal{A}$ is, as it should be, (anti-) linear anti-involution which means $(a^*)^* = a$ and $(ab)^* = b^*a^*$. By the way, the very concept of involution can be generalized so that it becomes adapted to ternary structures. A ternary involution should satisfy $((a^*)^*)^* = a$, as an example, we can introduce the

operation $*$ such that $[abc] = [b^*c^*a^*]$. In some applications, an important role is played by ternary algebras with a different associativity law:

$$[[abc]de] = [a[dcbe]] = [ab[cde]]. \quad (1)$$

Such associativity is sometimes called “type B -associativity” or “2nd kind” [3]. In the case of ternary \star -algebras both types of associativity are related to each other. Assuming that $(\mathcal{A}, [\]_*, *)$ is a ternary \star -algebra, one can introduce another ternary multiplication $[\]_*$ such that

$$[abc]_* \stackrel{def}{=} [ab^*c], \quad \forall a, b, c \in \mathcal{A}.$$

The algebra $(\mathcal{A}, [\]_*, *)$ becomes an associative ternary \star -algebra of B -type. The converse statement is also true: any ternary \star -algebra of B -type gives rise to a standard ternary \star -algebra. Observe that in the case of algebras over the field of complex numbers one has to assume anti-linearity of ternary multiplication in the middle factor instead of linearity.

Example Any Hilbert or symmetric scalar product (i.e. metric) vector space \mathcal{H} bears a canonical structure of ternary algebra of B -type with ternary multiplication defined as follows:

$$\{a b c\} = \langle a, b \rangle c$$

induced by scalar multiplication \langle, \rangle in \mathcal{H} . It is not a ternary \star -algebra.

In the finite-dimensional case, when we replace the Hilbert space with a metric vector space, in a given basis $\{e_k\}$, $k, m = 1, 2, \dots, N$ we can define a non-degenerate metric $g_{ik} = \langle e_i, e_k \rangle$. Then the Clifford algebra generated by the elements C_i satisfying

$$C_i C_j + C_j C_i = 2g_{ij} \mathbf{1}$$

provides an appropriate associative algebra which can serve as a representation of the ternary product $\{e_i e_j e_k\} = \langle e_i, e_j \rangle e_k$ as follows:

$$\{C_i C_j C_k\} = \frac{1}{2} (C_i C_j + C_j C_i) C_k$$

Obviously, there are two other possible choices of ternary product in a metric (or Hilbert) space, corresponding to cyclic permutations of three factors:

$$\{a b c\}' = \langle b, c \rangle a, \quad \{a b c\}'' = \langle c, a \rangle b$$

In a finite-dimensional case, one may define the most general ternary product of this type as a linear combination of these three, i.e.

$$\{e_i e_j e_k\} = \sum_{l,m,n} M_{ijk}^{lmn} \langle e_l, e_m \rangle e_n = \rho_{ijk}^n e_n \quad (2)$$

with tensors M_{ijk}^{lmn} symmetric in first two upper indices l, m . The four-index tensor ρ_{ijk}^n plays the role of structure constants of our ternary algebra. It is easy to prove that it is impossible to impose strong associativity on such a product, because the set of equations it would imply on the coefficients of the tensor M_{ijk}^{lmn} is strongly over-determined (see ([20]) for example).

As in the usual algebraic case, we can impose particular symmetries on the ternary product, defining a new product displaying a representation property with respect to the permutations of its three lower indices, e.g. by requiring the total symmetry:

$$\begin{aligned} \{e_i e_j e_k\}_{\text{sym}} &= \{e_i e_j e_k\} + \{e_j e_k e_i\} + \{e_k e_i e_j\} = \\ &\langle e_i, e_j \rangle e_k + \langle e_j, e_k \rangle e_i + \langle e_k, e_i \rangle e_j \end{aligned} \quad (3)$$

Other choices are possible; for example, a Z_3 -generalization of the commutator in associative binary algebras, which generates non-associative Lie algebras, can be introduced as follows:

$$\{e_i e_j e_k\}_q = \{e_i e_j e_k\} + q \{e_j e_k e_i\} + q^2 \{e_k e_i e_j\} \quad (4)$$

with q one of the primitive third roots of unity, $q = e^{\frac{2i\pi}{3}}$, satisfying $q^3 = 1$ and $q + q^2 + q^3 = 0$.

This algebra is particularly simple in dimension two, when there are only two basis vectors. Then the "ternary structure constants" ρ_{jkm}^i are as follows:

$$\begin{aligned} \rho_{111}^i &= \rho_{222}^i = 0; \\ \rho_{221}^1 &= q \rho_{212}^1 = q^2 \rho_{122}^1 = 1; \quad \rho_{112}^2 = q \rho_{121}^2 = q^2 \rho_{211}^2 = 1 \\ \rho_{221}^2 &= q \rho_{212}^2 = q^2 \rho_{122}^2 = 0; \quad \rho_{112}^1 = q \rho_{121}^1 = q^2 \rho_{211}^1 = 0 \end{aligned} \quad (5)$$

Besides their particular symmetry, the coefficients ρ_{jkm}^i possess another interesting property akin to the representation property of antisymmetric structure constants of usual Lie algebras. Let us introduce the following

ternary composition law for these coefficients, which can be also named "cubic matrices", with regard to their lower indices:

$$(\rho^i * \rho^j * \rho^k)_{prs} = \sum_{nmt} \rho_{npm}^i \rho_{mrt}^j \rho_{tsn}^k \quad (6)$$

Then, introducing the same Z_3 -skew-symmetric product as

$$\{\rho^i \rho^j \rho^k\}_q = (\rho^i * \rho^j * \rho^k) + q(\rho^j * \rho^k * \rho^i) + q^2(\rho^k * \rho^i * \rho^j), \quad (7)$$

we can easily check that

$$\{\rho^i \rho^j \rho^k\}_q = \sum_m \rho_m^{ijk} \rho^m \quad (8)$$

which provides us with a faithful representation of our ternary algebra.

As in the usual case, we are interested to know whether such an algebra can be also represented by certain combinations of ternary products in an ordinary (i.e. binary) associative algebra, playing the role of an enveloping algebra. It is easy to see that the answer is positive. In the above example, the two generators of non-associative ternary algebra with Z_3 -skew-symmetric product can be represented by any two Pauli matrices multiplied by the factor $i/2$. One can check that the matrices $\tau_i = \frac{i}{2} \sigma_i$, $i = 1, 2$ satisfy

$$\sigma_i \sigma_j \sigma_k + q \sigma_j \sigma_k \sigma_i + q^2 \sigma_k \sigma_i \sigma_j = \sum_m \rho_{ijk}^m \sigma_m \quad (9)$$

2.2 Universal envelope of ternary algebra

In order to construct an universal envelope of ternary algebra one should consider the structure of ternary algebras in more detail. In the classical (binary) case, algebras defined by generators and relations between them play important role in concrete applications. It suffices to mention that the well-known Grassmann and Clifford algebras can be defined in this way. Let us briefly summarize this approach (see e.g. [2]). First, we recall that the tensor algebra

$$TV = \bigoplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

of a given vector space V is a free algebra with n -generators, where $n = \dim V$. Thus for any subset $S \subset TV$ one can construct a two-sided ideal J_S generated by S and the quotient algebra

$$\mathcal{A}_S = TV/J_S .$$

Here V is called a space of generators, S is a set of generating relations. Conversely, by the well-known theorem (see e.g. [2]) any (unital) algebra with n - generators can be obtained in this way. Notice that a non-unital free algebra can be defined as

$$T'V = \oplus_{k=1}^{\infty} V^{\otimes k} = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

Much in the same way, for any vector space V one can construct a free ternary algebra generated by V . To this aim we define

$$T^{\text{odd}}V = \oplus_{k=0}^{\infty} V^{\otimes(2k+1)} = V \oplus V^{\otimes 3} \oplus V^{\otimes 5} \oplus \dots \quad (10)$$

as a ternary algebra with a ternary multiplication:

$$[uvw]_{\otimes} = u \otimes v \otimes w, \quad \forall u, v, w \in T^{\text{odd}}V. \quad (11)$$

Observe that $T^{\text{odd}}V$ is not a trivial ternary algebra, however it is a ternary subalgebra in the trivial ternary algebra TV (as well as in $T'V$).

$T^{\text{odd}}V$ plays the role of free ternary algebra in the following way. Let $(\mathcal{A}, [\])$ be a ternary algebra, V any vector space. Then for any linear map $\varphi : V \rightarrow \mathcal{A}$, there exists its unique lift $\tilde{\varphi} : T^{\text{odd}}V \rightarrow \mathcal{A}$, which is a homomorphism of ternary algebras, such that $\varphi = \tilde{\varphi} \circ \mu$, which means that the following diagram

$$\begin{array}{ccc} & & T^{\text{odd}}V \\ & \nearrow \mu & \downarrow \tilde{\varphi} \\ V & \xrightarrow{\varphi} & \mathcal{A} \end{array}$$

is commutative. Here μ denotes the canonical embedding $\mu : V \hookrightarrow T^{\text{odd}}V$.

In particular, if φ is an embedding and $\tilde{\varphi}$ is an epimorphism, then

$$\mathcal{A} \cong T^{\text{odd}}V / \text{Ker}(\tilde{\varphi})$$

i.e. the algebra \mathcal{A} becomes isomorphic to the quotient algebra $T^{\text{odd}}V / \text{Ker}(\tilde{\varphi})$. Quite obviously, $\text{Ker}\tilde{\varphi}$ is a ternary ideal in $T^{\text{odd}}V$. The simplest example is provided tautologically by the fact that

$$\mathcal{A} \cong T^{\text{odd}}\mathcal{A} / \text{gen} \langle a \otimes b \otimes c - [abc] \rangle,$$

where $\text{gen} \langle a \otimes b \otimes c - [abc] \rangle$ denotes a ternary ideal generated by elements $\{a \otimes b \otimes c - [abc] : a, b, c \in \mathcal{A}\}$.

Any n -nary algebra can be embedded into a binary one [4]. Here we are particularly interested in the ternary case [3]. For given ternary algebra \mathcal{A} one can define a \mathbb{Z}_2 -graded vector space

$$\mathcal{U}_{\mathcal{A}} = \mathcal{A}_1 \oplus \mathcal{A}_0,$$

where $\mathcal{A}_1 = \mathcal{A}$ is an odd part. The even subspace \mathcal{A}_0 of $\mathcal{U}_{\mathcal{A}}$ is assumed to be the quotient vector space

$$\mathcal{A}_0 = (\mathcal{A} \otimes \mathcal{A}) / \text{span} \langle [xyz] \otimes w - x \otimes [yzw] \rangle$$

where $\text{span} \langle [xyz] \otimes w - x \otimes [yzw] \rangle$ denotes a vector subspace of $\mathcal{A} \otimes \mathcal{A}$ spanned by elements $\{[xyz] \otimes w - x \otimes [yzw] : x, y, z, w \in \mathcal{A}\}$. Let $a \circledast b$ denote the equivalence class of the element $a \otimes b \in \mathcal{A} \otimes \mathcal{A}$. Now we are in a position to define the multiplication $\overline{\circledast}$ between elements from $\mathcal{U}_{\mathcal{A}}$ by the following:

$$\begin{aligned} a \overline{\circledast} b &\stackrel{\text{def}}{=} a \circledast b; \\ (a \circledast b) \overline{\circledast} c &= a \overline{\circledast} (b \circledast c) \stackrel{\text{def}}{=} [abc]; \\ (a \circledast b) \overline{\circledast} (c \circledast d) &\stackrel{\text{def}}{=} [abc] \circledast d = a \circledast [bcd] = a \overline{\circledast} ((b \circledast c) \overline{\circledast} d) = (a \overline{\circledast} (b \circledast c)) \overline{\circledast} d. \end{aligned}$$

In this way, we have obtained a \mathbb{Z}_2 -graded algebra, since

$$\mathcal{A}_i \circledast \mathcal{A}_j \subset \mathcal{A}_{i+j \pmod{2}}.$$

It is easy to see that this binary, nonunital algebra is associative. The initial ternary algebra $\mathcal{A} \equiv \mathcal{A}_1$ becomes a ternary subalgebra in the trivial ternary algebra $\mathcal{U}_{\mathcal{A}}$. Of course, \mathcal{A}_0 is a (binary) algebra which is also a subalgebra of $\mathcal{U}_{\mathcal{A}}$, and \mathcal{A} becomes a \mathcal{A}_0 -bimodule. Further on, to simplify the notation, we shall use the same symbol in order to denote the equivalence class $a \circledast b \in \mathcal{A}_0$ corresponding to the elements $a, b \in \mathcal{A}$, and for the multiplication $\overline{\circledast}$ in $\mathcal{U}_{\mathcal{A}}$.

In other words, any ternary algebra can be extended to a binary one, i.e. a ternary algebra is a ternary subalgebra in some associative binary algebra. Furthermore, if $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a ternary homomorphism of \mathcal{A} into an associative algebra \mathcal{B} , i.e. \mathcal{A} is a ternary subalgebra in a binary algebra \mathcal{B} , then there exists one and only one (binary) algebra homomorphism $\tilde{\phi} : \mathcal{U}_{\mathcal{A}} \rightarrow \mathcal{B}$ such that $\phi = \tilde{\phi} \circ \iota$, where ι is the canonical embedding of \mathcal{A} into $\mathcal{U}_{\mathcal{A}}$, i.e. the following diagram:

$$\begin{array}{ccc} & & \mathcal{U}_{\mathcal{A}} \\ & \nearrow \iota & \downarrow \tilde{\phi} \\ \mathcal{A} & \xrightarrow{\phi} & \mathcal{B} \end{array}$$

is commutative, and this universal property characterizes $(\iota, \mathcal{U}_{\mathcal{A}})$ up to an isomorphism. For example, $\mathcal{U}_{T^{\text{odd}}V} = T'V$, i.e.

$$T'V = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus V^{\otimes 4} \oplus \dots = T^{\text{odd}}V \oplus T^{\text{even}}V,$$

is the enveloping algebra, in which the ternary algebra $(T^{\text{odd}}V, [\]_{\otimes})$ can be embedded.

2.3 Tri-modules over ternary algebras.

The concept of tri-module is a particular case of the concept of module over an algebra over an operad defined in [11]. In the more general context of n -ary algebras it was then considered in [12]. Here, a structure of tri-module over a ternary algebra \mathcal{A} is simply defined on a vector space \mathcal{M} by the following three linear mappings called

$$\begin{aligned} \text{left} \quad & [\]_L : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{M}, \\ \text{right} \quad & [\]_R : \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{M}, \\ \text{and central} \quad & [\]_C : \mathcal{A} \otimes \mathcal{M} \otimes \mathcal{A} \rightarrow \mathcal{M} \end{aligned}$$

multiplication, respectively (see also [3, 16]). They are assumed to satisfy the following compatibility conditions

$$[ab[cdm]_L]_L = [[abc]dm]_L = [a[bcd]m]_L, \quad (12)$$

$$[[mab]_Rcd]_R = [ma[bcd]]_R = [m[abc]d]_R, \quad (13)$$

$$[a[b[cmx]_Cy]_Cz]_C = [[abc]m[xyz]]_C, \quad (14)$$

$$[a[bcm]_Ld]_C = [ab[cmd]_C]_L = [[abc]md]_C, \quad (15)$$

$$[a[mbc]_Rd]_C = [[amb]_Ccd]_R = [am[bcd]]_C, \quad (16)$$

$$[[abm]_Lcd]_R = [ab[mcd]_R]_L = [a[bmc]_Cd]_C, \quad (17)$$

$$\forall a, b, c, d, x, y, z \in \mathcal{A}, m \in \mathcal{M}$$

In the case of tri-module \mathcal{M} over an algebra \mathcal{A} of type B the conditions (12), (13), (17) remain unchanged while (14), (15), (16) have to be replaced correspondingly by

$$[a[b[cmx]_Cy]_Cz]_C = [[ayc]m[xbz]]_C, \quad (18)$$

$$[a[cbm]_Ld]_C = [[amb]_Ccd]_R, \quad (19)$$

$$[a[mbc]_Rd]_C = [ab[cmd]_C]_L, \quad (20)$$

$$\forall a, b, c, d, x, y, z \in \mathcal{A}, m \in \mathcal{M}.$$

Much in the same way as binary algebra is a trivial ternary algebra, the notion of tri-module generalizes the notion of bimodule. More exactly, one has

REMARK. Let \mathcal{A} be a (binary) algebra and \mathcal{M} a bimodule over it. Thus defining $[abm]_L = a \cdot (b \cdot m) = (a \cdot b) \cdot m$, $[amb]_C = a \cdot (m \cdot b) = (a \cdot m) \cdot b$ and $[mab]_R = m \cdot (a \cdot b) = (m \cdot a) \cdot b$ we see that \mathcal{M} becomes a tri-module over the same algebra considered as a trivial ternary algebra.

Analogously, we can obtain an enveloping module $\mathcal{U}_{\mathcal{M}}$ over the enveloping algebra $\mathcal{U}_{\mathcal{A}}$ of ternary module \mathcal{M} over ternary algebra \mathcal{A} . Let us denote by $\mathcal{U}_{\mathcal{M}}$ a \mathbb{Z}_2 -graded vector space

$$\mathcal{U}_{\mathcal{M}} = \mathcal{M}_1 \oplus \mathcal{M}_0, \quad (21)$$

where the odd part $\mathcal{M}_1 \equiv \mathcal{M}$. The even part is defined as a quotient vector space

$$\mathcal{M}_0 = (\mathcal{A} \otimes \mathcal{M} \oplus \mathcal{M} \otimes \mathcal{A}) / \text{lin} \langle S \rangle,$$

where S is a set of elements in $\mathcal{A} \otimes \mathcal{M} \oplus \mathcal{M} \otimes \mathcal{A}$ ($a, b, c \in \mathcal{A}, m \in \mathcal{M}$.)

$$\begin{aligned} & [abc] \otimes m - a \otimes [bcm]_L, \\ & [abm]_L \otimes c - a \otimes [bmc]_C, \\ & [amb]_C \otimes c - a \otimes [mbc]_R, \\ & [mab]_R \otimes c - m \otimes [abc], \end{aligned}$$

which generate the subspace $\text{lin} \langle S \rangle$.

As previously, denote by $a \otimes m$ or $m \otimes a$ the corresponding equivalence classes, elements of \mathcal{M}_0 . Define left and right multiplication $\overline{\otimes}$ between elements from $\mathcal{U}_{\mathcal{A}}$ and those from $\mathcal{U}_{\mathcal{M}}$ in the following way:

$$\begin{aligned} a \overline{\otimes} m &\stackrel{def}{=} a \otimes m; \\ m \overline{\otimes} a &\stackrel{def}{=} m \otimes a; \\ (a \otimes b) \overline{\otimes} m &= a \overline{\otimes} (b \otimes m) \stackrel{def}{=} [abm]_L; \\ m \overline{\otimes} (c \otimes d) &= (m \otimes c) \overline{\otimes} d \stackrel{def}{=} [mcd]_R; \\ a \overline{\otimes} (m \otimes b) &\stackrel{def}{=} [amb]_C; \quad (a \otimes m) \overline{\otimes} b \stackrel{def}{=} [amb]_C; \\ \forall a, b, c, d, e, f \in \mathcal{A}, m \in \mathcal{M}. \end{aligned}$$

One can check the following properties of the action of algebra \mathcal{A} on module \mathcal{M}

$$\begin{aligned} [abc] \otimes m &= a \otimes [bcm]_L; \\ m \otimes [bcd] &= [mbc]_R \otimes d; \\ [abc] \otimes (m \otimes d) &= [ab[cmd]_C]_L; \\ (a \otimes m) \otimes [bcd] &= [[amb]_C cd]_R; \end{aligned}$$

$$\begin{aligned}
(a \otimes [bcd]) \otimes m &= ([abc] \otimes d) \otimes m = [abc] \otimes (d \otimes m) = [ab[cdm]_L]_L; \\
m \otimes ([cde] \otimes f) &= m \otimes (c \otimes [def]) = (m \otimes c) \otimes [def] = [[mcd]_{Ref}]_R; \\
(a \otimes [bcd]) \otimes (m \otimes e) &= ([abc] \otimes d) \otimes (m \otimes e) = [abc] \otimes [dme]_C; \\
(a \otimes m) \otimes ([bcd] \otimes e) &= (a \otimes m) \otimes (b \otimes [cde]) = [[amb]_C \otimes [cde]];
\end{aligned}$$

Thus $\mathcal{U}_{\mathcal{M}}$ becomes a \mathbb{Z}_2 -graded bimodule over $\mathcal{U}_{\mathcal{A}}$ since

$$\mathcal{A}_i \overline{\otimes} \mathcal{M}_j \subseteq \mathcal{M}_{i+j(mod 2)}, \quad \mathcal{M}_j \overline{\otimes} \mathcal{A}_i \subseteq \mathcal{M}_{i+j(mod 2)}, \quad i, j \in \{0, 1\}.$$

In particular, $\mathcal{M} \equiv \mathcal{M}_1$ and \mathcal{M}_0 are \mathcal{A}_0 -bimodules.

Further on, we shall use the same symbol \otimes to denote the equivalence class in $\mathcal{U}_{\mathcal{M}}$, its bimodule structure and for the multiplication in $\mathcal{U}_{\mathcal{A}}$.

Let us stress again that any bimodule over a (binary) algebra becomes automatically a trimodule over the same algebra considered as a trivial ternary algebra.

What we have shown above is that any trimodule is a sub-trimodule of some universal bimodule $\mathcal{U}_{\mathcal{M}}$ over $\mathcal{U}_{\mathcal{A}}$. Conversely, if \mathcal{N} is a \mathbb{Z}_2 -graded bimodule over $\mathcal{U}_{\mathcal{A}}$, then its odd part \mathcal{N}_1 is a trimodule over \mathcal{A} .

3 Universal differentiation of ternary algebra

A first order differential calculus (differential calculus in short) of ternary algebra \mathcal{A} is a linear map from ternary algebra \mathcal{A} into tri-module over it, i.e. $d : \mathcal{A} \rightarrow \mathcal{M}$, such that a ternary analog of the Leibniz rule takes place:

$$d([f g h]) = [df g h]_R + [f dg h]_C + [f g dh]_L, \quad \forall f, g, h \in \mathcal{A}. \quad (22)$$

In particular, if $\mathcal{M} = \mathcal{A}$, then we shall call so defined differential *ternary derivation* of \mathcal{A} . An interesting example is provided by

Example 1. *Ternary derivative in Hilbert (or metric) vector space.*

As we already noticed, any Hilbert space $(\mathcal{H}, \langle, \rangle)$ inherits a canonical ternary 2nd type associative structure given by $\{a b c\} = \langle a, b \rangle c$.

For a linear operator being a ternary derivation $D : \mathcal{H} \rightarrow \mathcal{H}$ one calculates:

$$D\{a b c\} = \{Da b c\} + \{a Db c\} + \{a b Dc\},$$

Now, taking into account that $D\{a b c\} = \langle a, b \rangle Dc = \{a b Dc\}$ it implies

$$\langle Da, b \rangle = - \langle a, Db \rangle \Rightarrow D^+ = -D, \quad \text{i.e.} \quad (iD)^+ = iD$$

i.e., that ternary derivations are in one-to-one correspondence with hermitian operators in \mathcal{H} . This makes possible a link with Quantum Mechanics, especially the version introduced by Nambu ([17]).

Let us refer again to the classical (binary) case. First order differential calculus from an algebra into bimodule can be automatically interpreted as a ternary differential calculus from trivial ternary algebra into a trivial tri-module over it. It can be easily seen from

$$d(fgh) = d((fg)h) = d(fg)h + fg d(h) = df gh + f dg h + fg dh .$$

The converse statement is, in general, not true. A ternary Leibniz rule for differential calculus from an algebra into bimodule does not necessarily imply, in case of non-unital algebras, the existence of a standard (binary) Leibniz rule. In particular, the set of ternary derivations of non-unital algebra should be an extension of the set of standard (binary) derivations.

Let $(\mathcal{A}, \mathcal{M}, d)$ be a our ternary differential calculus from ternary algebra into tri-module. By the Leibniz rule

$$\tilde{d}(a \otimes b) = (\tilde{d}a) \otimes b + a \otimes (\tilde{d}b)$$

it can be uniquely extended to a 0-degree differential $\tilde{d} : \mathcal{U}_{\mathcal{A}} \rightarrow \mathcal{U}_{\mathcal{M}}$, in a way which ensures commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{d} & \mathcal{M} \\ \downarrow \iota & & \downarrow \iota' \\ \mathcal{U}_{\mathcal{A}} & \xrightarrow{\tilde{d}} & \mathcal{U}_{\mathcal{M}} \end{array}$$

Conversely, any 0-degree first order differential calculus from $\mathcal{U}_{\mathcal{A}}$ into $\mathcal{U}_{\mathcal{M}}$, such that $\tilde{d}|_{\mathcal{A} \subset \mathcal{M}}$ gives rise to ternary \mathcal{M} -valued differential calculus on \mathcal{A} .

The universal first order differential calculus on non-unital algebras is well describe in [6, 7, 14]. Let us recall this construction shortly. Determine a vector space $\Omega_u^1(\hat{\mathcal{A}}) = \hat{\mathcal{A}} \oplus \hat{\mathcal{A}} \otimes \hat{\mathcal{A}}$, where $\hat{\mathcal{A}}$ is a non-unital (binary) algebra. Any element from $\Omega_u^1(\hat{\mathcal{A}})$ can be written in the form: $(a, b \otimes c)$, where $a, b, c \in \hat{\mathcal{A}}$. Define left and right multiplications by elements from $\hat{\mathcal{A}}$:

$$\begin{aligned} x(a, b \otimes c) &= (0, x \otimes a + xb \otimes c), \\ (a, b \otimes c)y &= (ay, -a \otimes y + b \otimes cy - bc \otimes y). \end{aligned}$$

In this way, $\Omega_u^1(\hat{\mathcal{A}})$ becomes a $\hat{\mathcal{A}}$ -bimodule since

$$(x(a, b \otimes c))y = x((a, b \otimes c)y).$$

Let $D : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}} \oplus \hat{\mathcal{A}} \otimes \hat{\mathcal{A}}$, $Da = (a, 0)$, $\forall a \in \hat{\mathcal{A}}$ be a canonical embedding. Because it satisfies the Leibniz rule:

$$D(a)b + aD(b) = (a, 0)b + a(b, 0) = (ab, a \otimes b) + (0, -a \otimes b) = (ab, 0) = D(ab),$$

D is a differential. We shall call it the universal differential for a given algebra $\hat{\mathcal{A}}$.

For unital algebras, there exists an alternative construction of $\Omega_u^1(\hat{\mathcal{A}})$ as a kernel of multiplication map [5] (see also [2]). Since our ternary algebras have no unit element, we can not use such construction here.

Our aim is to provide an analogous construction in the case of ternary algebra.

From a \mathbb{Z}_2 -graded $\mathcal{U}_{\mathcal{A}}$ -bimodule $\Omega_u^1(\mathcal{U}_{\mathcal{A}}) = \mathcal{U}_{\mathcal{A}} \oplus \mathcal{U}_{\mathcal{A}} \otimes \mathcal{U}_{\mathcal{A}}$, let us extract its odd subspace $\mathcal{A} \oplus \mathcal{A}_0 \otimes \mathcal{A} \oplus \mathcal{A} \otimes \mathcal{A}_0$ with elements

$$(a, \beta \otimes b, c \otimes \gamma), \forall a, b, c, \beta, \gamma \in \mathcal{A}_0.$$

We shall denote it as $\Omega_T^1(\mathcal{A}) = \mathcal{A} \oplus \mathcal{A}_0 \otimes \mathcal{A} \oplus \mathcal{A} \otimes \mathcal{A}_0$. As we already know from our previous considerations, $\Omega_T^1(\mathcal{A})$ is a tri-module over \mathcal{A} . Thus we have defined the left, central and right ternary multiplications

$$\begin{aligned} [xy(a, \beta \otimes b, c \otimes \gamma)]_L &= (0, (x \otimes y) \otimes a + (x \otimes [y\beta]) \otimes b, [xyc] \otimes \gamma); \\ [x(a, \beta \otimes b, c \otimes \gamma)y]_C &= (0, -(x \otimes a) \otimes y - ([x\beta] \otimes \gamma) \otimes y + (x \otimes c) \otimes [\gamma y] - \\ &\quad (x \otimes [c\gamma]) \otimes y, x \otimes (a \otimes y) + [x\beta] \otimes (b \otimes y)); \\ [(a, \beta \otimes b, c \otimes \gamma)xy]_R &= \\ &= ([axy], \beta \otimes [bxy], -a \otimes (x \otimes y) - [\beta b] \otimes (x \otimes y) + c \otimes ([\gamma x] \otimes y) - [c\gamma] \otimes (x \otimes y)). \end{aligned} \tag{23}$$

The canonical embedding $D : \mathcal{A} \rightarrow \Omega_T^1\mathcal{A}$:

$$D(a) = (a, 0, 0), \quad \forall a \in \mathcal{A}. \tag{24}$$

defines a ternary differential (22). In fact, one has

$$\begin{aligned} [(a, 0, 0)bc]_L + [a(b, 0, 0)c]_C + [ab(c, 0, 0)]_R &= \\ ([abc], 0, a \otimes (b \otimes c)) + (0, -(a \otimes b) \otimes c, -a \otimes (b \otimes c)) + (0, (a \otimes b) \otimes c, 0) &= \\ ([abc], 0, 0). \end{aligned}$$

This ternary differential calculus is universal because for any trimodule E and any ternary E -valued differential calculus $d : \mathcal{A} \rightarrow E$, there exists one

and only one covering trimodule homomorphism $\tilde{\varphi}_d$ such that $d = \tilde{\varphi}_d \circ D$, i.e. the following diagram

$$\begin{array}{ccc} & \Omega_T^1(\mathcal{A}) & \\ D \nearrow & \downarrow \tilde{\varphi}_d & \\ \mathcal{A} & \xrightarrow{d} & E \end{array}$$

is commutative. Moreover, if the trimodule E is spanned by the elements $d\mathcal{A}$, $[\mathcal{A} \ d\mathcal{A} \ \mathcal{A}]_C$ and $[d\mathcal{A} \ \mathcal{A} \ \mathcal{A}]_R$, then $\tilde{\varphi}_d$ is an epimorphism and $E = \Omega_T^1(\mathcal{A})/\text{Ker}(\tilde{\varphi}_d)$.

In this way, the problem of classification of all first order differential calculi over \mathcal{A} can be translated into the problem of classification of all sub-trimodules in $\Omega_T^1(\mathcal{A})$. Remember that $\Omega_T^1(\mathcal{A})$ is an odd part of $\Omega_u^1(\mathcal{U}_{\mathcal{A}})$ and our ternary differential (24) is, in fact, a restriction of the universal differential.

As it is well known [6, 7, 14] the bimodule $\Omega_u^1(\mathcal{U}_{\mathcal{A}})$ extends, by means of the graded Leibniz rule, to the universal graded differential algebra with $d^2 = 0$. This leads to higher order differential calculi. Another universal extension with $d^N = 0$, still for the case of binary (unital) algebras, has been considered in [8, 9]. These universal extensions have been provided by means of the q -Lebniz rule, for q being primitive N -degree root of the unity, i.e. $q = e^{\frac{2\pi i}{N}}$ (see also [13] in this context). However, the so-called N -ary case ($d^N = 0$) seems also to be specially well adopted for N -ary algebras. Various constructions of higher order differentials for ternary algebras will be a subject of our future investigation.

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